TWO MODELING PROBLEMS FOR THE MOTION OF AN AGGRESSIVE LIQUID IN A POROUS MEDIUM

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During the filtration of a liquid through a porous medium, the liquid being filtered may enter into a reaction with the medium, causing a change in the pore volume. Examples of this are the dissolution of salts in soil, the leaching of rock, etc. The problems discussed in this paper—those of an incompressible liquid entering a circular tube (or capillary) or of a slit reacting with its walls—are of some practical interest and may also serve as the simplest models of these processes.

1. **Motion of liquid in a tube.** We let $Q(x,t)$ be the volume flow rate of the liquid, $R(x,t)$ the inner tube radius, $C(x,t)$ the liquid concentration, C_* the saturation concentration, and t the time; we let the x-axis run parallel to the tube. We assume that a first-order reaction is occurring between the liquid and the material making up the tube walls, causing an increase in the tube radius. By "first-order reaction," we mean a reaction whose rate is proportional to the surface area of the walls in contact with the liquid and to the difference between the solution concentration and the saturation concentration [1].

The tube radius is assumed to increase by dR during a time dt over a length dx. Then, denoting the density of the tube walls by ρ_1, we have

$$\rho_1 [\pi (R + dR)^2 dx - \pi R^2 dx] = A 2\pi R dx (C_* - C) dt ,$$

from which we find the kinetic equation for the reaction:

$$\frac{\partial R}{\partial t} = \frac{A}{\rho_1} (C_* - C). \qquad (1.1)$$

Here A is a constant (the salt yield in the case of a dissolution reaction).

The mass-conservation equations for the dissolved substance and for the mass of the moving solution are

$$-\frac{\partial}{\partial x}(QC) + 2\pi AR (C_* - C) = \frac{\partial}{\partial t}(\pi R^2 C), \qquad -\frac{d}{\partial x}(Q\rho) = \frac{\partial}{\partial t}(\pi R^2 \rho), \qquad (1.2)$$

where ρ is the solution density, which depends on its concentration. We can approximate this dependence by the linear dependence

$$\rho = \rho_0 (1 + \gamma c) \quad (c = C/C_*) . \qquad (1.3)$$

Here ρ_0 is the solvent (water) density, and γ is a constant. We introduce the dimensionless quantities

$$q = \frac{Q}{\pi Q_0}, \qquad a = 2A\frac{R_0^2}{Q_0}, \qquad v_1 = \frac{C_*}{2\rho_1}, \qquad \xi = \frac{ax}{R_0}, \qquad \tau = \frac{atQ_0}{R_0^3}, \qquad r = \frac{R}{R_0}$$

(here Q_0 = const is the flow rate at the entrance cross section of the tube, and R_0 = const is the initial tube radius, i. e., the radius at t = 0).

To find the unknown functions $c(\xi, \tau)$, $r(\xi, \tau)$, $q(\xi, \tau)$, we write the system of equations

$$r^2 \frac{\partial c}{\partial \tau} + q \frac{\partial c}{\partial \xi} = r (1 - c)(1 + \gamma c),$$
$$\frac{\partial r}{\partial \tau} = v_1 (1 - c), \qquad \frac{\partial q}{\partial \xi} = -r (1 - c)(2v_1 + \gamma) . \qquad (1.4)$$

If the liquid has not entered the tube before t = 0, or a solution of saturation concentration is in it, and solvent supply at some flow rate Q_0 begins at t > 0, then the boundary conditions for systems (1.4) are

$$q = 1, \quad c = 0; \quad \xi = 0, \quad \tau > 0; \quad r = 1; \quad \xi = \xi_* (\tau), \quad \tau \geqslant 0. \tag{1.5}$$

The function $\xi_*(\tau)$ should satisfy the kinematic condition

$$\frac{d\xi_*}{d\tau} = \frac{q(\xi_*, \tau)}{r^2(\xi_*, \tau)} = q(\xi_*, \tau) \qquad (\xi_*(0) = 0) \tag{1.6}$$

(the characteristic equation for system (1.4)), and is sought along with the functions $c(\xi, \tau)$, $r(\xi, \tau)$, and $q(\xi, \tau)$, defined in the region

$$\xi < \xi_*(\tau), \quad \tau > 0.$$

Problem (1.4)–(1.6) is nonlinear. To simplify it, we linearize it by means of the small-perturbation method. We set

$$r = 1 + r', \; c = c', \; q = 1 + q', \tag{1.7}$$

where the increments r', c', and q' are small, so that their products and squares are negligible.

In addition, since the solution concentration density is assumed low, and the increase in the solution density with increasing concentration is generally not large, we assume here that $\gamma = 0$.

Substituting (1.7) into the first two equations of system (1.4), introducing the new independent variables $x_1 = \xi$ and $x_2 = \tau - \xi$, and discarding small quantities of higher orders, we find

$$\frac{\partial c}{\partial x_1} = r' + 1 - c', \qquad \frac{\partial r'}{\partial x_2} = v_1 (1 - c'). \tag{1.8}$$

Setting $q \approx 1$ in condition (1.6), we find $\xi_*(\tau) \equiv \tau$ the boundary conditions (1.5) for the functions $c'(x_1, x_2)$ and $r'(x_1, x_2)$ become

$$c' = 0, \; x_1 = 0, \; x_2 \geqslant 0; \; r' = 0, \; x_2 = 0, \; x_1 \geqslant 0. \tag{1.9}$$

We let

$$c_1(x_1, p) = \int_0^\infty c' e^{-px_2} dx_2, \qquad r_1 = \int_0^\infty r' e^{-px_2} dx_2$$

be the Laplace transforms of the functions c' and r'. Multiplying Eqs. (1.8) by the kernel of the transform, integrating over x_2 from 0 to ∞, and using the second condition of (1.9), we find

$$\frac{dc_1}{dx_1} = r_1 + \frac{1}{p} - c_1, \qquad pr_1 = v_1\left(\frac{1}{p} - c_1\right).$$

Eliminating r_1 and integrating the resulting differential equation with account for the first condition (1.9), we find

$$c_1(x_1, p) = p^{-1}\{1 - \exp[-x_1(1 + v_1/p)]\}.$$

Performing the inverse transformation [2], we find

$$c = c'(x_1, x_2) = 1 - e^{-x_1} J_0(2\sqrt{v_1 x_1 x_2}).$$

From the first equation in system (1.8), we find

$$r' = -1 + c' + \frac{\partial c'}{\partial x_1} = e^{-x_1}\sqrt{\frac{v_1 x_2}{x_1}} J_1(2\sqrt{v_1 x_1 x_2}).$$

Here $J_0(z)$ and $J_1(z)$ are zeroth- and first-order Bessel functions of the first kind, respectively.

Accordingly, the solution of the linearized problem is written

$$c(\xi, \tau) = 1 - e^{-\xi} J_0 [2 \sqrt{v_1 \xi (\tau - \xi)}],$$
$$r(\xi, \tau) = 1 + e^{-\xi} \sqrt{v_1 (\tau - \xi)/\xi} \, J_1 [2 \sqrt{v_1 \xi (\tau - \xi)}],$$
$$q(\xi, \tau) = 1 - 2v_1 \int_0^{\xi} e^{-\xi} \sqrt{v_1 (\tau - \xi)/\xi} \, J_1 [2 \sqrt{v_1 \xi (\tau - \xi)}] \, d\xi.$$

In particular, we have

$$r(0, \tau) = \lim_{\xi \to 0} [1 + e^{-\xi} \sqrt{v_1 (\tau - \xi)/\xi} \sqrt{v_1 \xi (\tau - \xi)}] = 1 + v_1 \tau.$$

The solid and dashed curves in Fig. 1 show the functions $r'(\xi, \tau)$ and $c(\xi, \tau)$, respectively, for $\tau = 1, 2, 3$ and $\nu_1 = 0.1$.

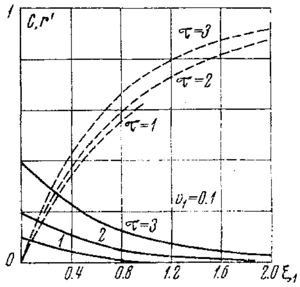

Fig. 1

2. **Motion of liquid within an infinite slit.** We let 2D be the slit thickness and $Q_1 = Q/2$ be half the flow rate per unit slit length. Then the equations for the conservation of mass of the soluble substance in the solid phase and in the solution, and that for the solution moving within the slit are

$$\rho_1 \frac{\partial D}{\partial t} = A(C_* - C),$$
$$-\frac{\partial}{\partial x}(Q_1 C) + A(C_* - C) = \frac{\partial}{\partial t}(DC), \tag{2.1}$$
$$-\frac{\partial}{\partial x}(Q_1 \rho) = \frac{\partial}{\partial t}(D\rho).$$

Introducing the dimensionless variables $\xi = Ax/Q_0$, $\tau = At/D_0$, $c = C/C_*$, $r = D/D_0$, $v_2 = C_*/\rho_1$, $q = Q_1/Q_0$, $p = 1 - c$ we can convert (2.1), after some simple transformations, and with account of (1.3), to

$$\frac{\partial r}{\partial \tau} = v_2 p, \qquad q \frac{\partial p}{\partial \xi} + r \frac{\partial p}{\partial \tau} = -p(1 - \gamma + \gamma p), \qquad \frac{\partial q}{\partial \xi} = -(v_2 + \gamma) p. \tag{2.2}$$

Here Q_0 is half the solvent flow rate across the cross section having abscissa $x = 0$, and $2D_0$ is the initial slit opening. The boundary conditions for system (2.2) are

$$p = q = 1, \ \xi = 0; \ r = 1, \ \xi = \xi_*(\tau), \tag{2.3}$$

where $\xi_*(\tau)$ satisfies Eq. (1.6).

In this case, the functions p and q, defined in the region $0 < \xi < \xi_*$, $\tau > 0$, do not depend explicitly on the variable τ. Setting $\partial p/\partial \tau = 0$, we find from system (2.2) that

$$q \frac{dp}{dq} = \frac{1 - \gamma + \gamma p}{v_2 + \gamma}, \qquad r(\xi, \tau) - r(\xi_*, \tau) = v_2 p (\tau - \tau_*).$$

Hence, using (2.3), we find

$$1 - \gamma + \gamma p = q^{\delta},$$
$$r = 1 + v_2 p (\tau - \tau_*) \ (\delta = \gamma/(v_2 + \gamma)). \tag{2.4}$$

Here $\tau_* = \tau_*(\xi)$ is the function inverse to the function $\xi_*(\tau)$. Substituting the value of p from (2.4), we convert the third equation in system (2.2) to

$$\delta \frac{dq}{d\xi} = 1 - \gamma - q^{\delta}.$$

Hence, using (2.3), we find

$$\xi = \delta \int_1^q \frac{dq}{1 - \gamma - q^{\delta}} = \delta I_1 (\gamma, \delta, q). \qquad (2.5)$$

In particular, it follows from (2.5) that

$$\lim q(\xi) = q_{\infty} = (1 - \gamma)^{1/\delta}, \quad \xi \to \infty;$$

i. e., at a sufficiently large distance from the slit, the liquid flow rate decreases, tending toward a constant value $q_{\infty} < 1$. Rewriting (1.6) as

$$\frac{d\tau_*}{d\xi} = \frac{1}{q} \qquad (\tau_*(0) = 0)$$

and integrating, we find

$$\tau_* = \int_0^{\xi} \frac{d\xi}{q} = \int_1^q \frac{d\xi}{dq}\frac{dq}{q} = \delta \int_1^q \frac{dq}{q(1 - \gamma - q^{\delta})} = \delta I_2 (\gamma, \delta, q).$$

Accordingly, the solution of this problem is reduced to quadratures and can be solved numerically in the general case, for arbitrary γ and $\delta \neq 0$.

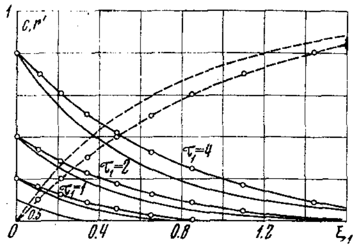

Fig. 2

We consider two particular examples.

Example 1. We set $v_2 = \gamma$; then $\delta = 0.5$, and the calculation of the integrals leads to the dependences

$$q = (1 - \gamma + \gamma p)^2 = (1 - \gamma c)^2,$$

$$\xi = \gamma(1 - p) - (1 - \gamma)\ln p = \gamma c - (1 - \gamma)\ln(1 - c),$$

$$\tau_* = \frac{1}{1 - \gamma}\ln\frac{1 - \gamma c}{1 - c}, \quad r = 1 + \gamma(1 - c)\left(\tau - \frac{1}{1 - \gamma}\ln\frac{1 - \gamma c}{1 - c}\right).$$

Example 2. We set $\gamma = 0$, $v_2 \neq 0$; then $\delta = 0$, and

$$q\,dp/d\xi = -p, \quad dq/d\xi = -v_2 p.$$

Hence $q\,dp/dq = 1/v_2$ ($p = 1$, $q = 1$) and

$$q = \exp[v_2(p - 1)] = \exp(-v_2 c).$$

Accordingly, we have

$$\exp[v_2(p - 1)]dp/d\xi = -p;$$

Integrating this with account for (2.3), we find

$$\xi = e^{-v_2}\,[\mathrm{Ei}\,(-v_2) - \mathrm{Ei}\,(-v_2 p)] \qquad \left(\mathrm{Ei}\,(-x) = -\int\limits_x^\infty \frac{e^{-t}}{t}\,dt\right).$$

By analogy with the preceding example, we find

$$\tau_* = -\ln p, \qquad r = 1 + v_2 p\,(\tau + \ln p)\,.$$

The solid and dashed curves in Fig. 2 show the functions $r' = r - 1$ and c, respectively, for $v_2 = 0.2$. The circles denote curves corresponding to Example 1. In particular, these graphs show that the assumption $\gamma = 0$ leads to values of c which are slightly too high and values of r' which are slightly too low.

REFERENCES

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